Lecture 2

1. Summary of Lecture 1: Technology

a. Describing technologies

- The firm's production decisions can be represented as a "netput" vector $\mathbf{y} = \{y_1, y_2, \dots , y_n\}$. The set of technologically feasible production plans is called the *production possibility set* and is denoted Y.

- If we only have one output the *production function* $f(\mathbf{x})$ describes the maximum level of output that can be obtained for a given vector of inputs $\mathbf{x}$. It constitutes the border of the production possibility set. In the case of several outputs the production set can be described by a *transformation function* $T(\mathbf{y})$ implicitly defined by $Y=\{\mathbf{y}\in\mathbb{R}^n: T(\mathbf{y})\leq 0\}$ and such that $T(\mathbf{y}) = 0$ only if the production plan is efficient. The set of efficient production plans is called the *transformation frontier*.

- The *input requirement set for* y, V(y), is the set of inputs $\mathbf{x}$ sufficient to produce y, i.e., $V(y)=\{\mathbf{x}\in\mathbb{R}^{n-1}: (y,-\mathbf{x}) \in Y\}$. Formally, input combinations $\mathbf{x}$ that yield the same (maximum) level of output y, define an *isoquant* $Q(y)=\{\mathbf{x}\in\mathbb{R}^{n-1}: \mathbf{x}\in V(y) \text{ and } \mathbf{x}\notin V(y'), y'>y\}$.

Example: Cobb-Douglas, V, p.4.

b. Properties of technologies

- Possibility of inaction: $\mathbf{0}\in Y$;
- Free disposal: $\mathbf{y}\in Y$, $\mathbf{y'}\leq \mathbf{y}\Rightarrow\mathbf{y'}\in Y$;
- Convexity: $\mathbf{y}, \mathbf{y}'\in Y$, $t\mathbf{y}+(1-t)\mathbf{y}'\in Y$, for all $0\leq t\leq 1$;
- No free lunch: $\mathbf{y}\in Y$ and $\mathbf{y}\geq\mathbf{0}\Rightarrow \mathbf{y}=\mathbf{0}$;
- Additivity: $\mathbf{y}, \mathbf{y}'\in Y\Rightarrow\mathbf{y}+\mathbf{y}'\in Y$;
- Irreversibility: $\mathbf{y}\in Y$ and $\mathbf{y}\neq\mathbf{0}\Rightarrow -\mathbf{y}\notin Y$;
- Nonincreasing returns to scale: $\mathbf{y}\in Y\Rightarrow t\mathbf{y}\in Y$, for all $0\leq t\leq 1$;

- Nondecreasing returns to scale: $\mathbf{y} \in Y \Rightarrow t\mathbf{y} \in Y$, for all $1 \leq t$;
- Constant returns to scale (CRS): $\mathbf{y} \in Y \Rightarrow t\mathbf{y} \in Y$, for all $0 \leq t$.

c. Properties of technologies cont.

The *technical rate of substitution* (TRS) between two inputs is simply the slope of Q(y) and can be obtained by totally differentiating $f(\mathbf{x}) = y$. It is thus the ratio of the marginal products of the inputs in question: $\dfrac{dx_1}{dx_2} = -\dfrac{\frac{\partial f}{\partial x_1}}{\frac{\partial f}{\partial x_2}}$.

The *elasticity of substitution* σ tells us how much the ratio of inputs, change with a change in the *TRS,* along the isoquant. If the isoquant is very curved then a considerable change in the slope will only be associated with a minor change in the ratio of inputs: $\sigma = \dfrac{d\ln(\frac{x_1}{x_2})}{d\ln|TRS|}$.

There are two classes of production functions with the convenient property that the *TRS*s are independent of the scale of production (i.e., a proportional change of all inputs starting from any point on a certain isoquant leads to the same change in output): *homogeneous functions* and *homothetic functions*. A function f(.) is *homogeneous of degree k* if $f(tx)=t^k f(x)$, for all $t>0$. To assume that a function is homothetic is a weaker assumption than to assume homogeneity but it still preserves the central property. A function g(.) is *homothetic* if $g(x) = F(f(x))$ where f(.) is homogenous of degree 1 and $F' > 0$.

Note that:

- A linear homogenous function ($k = 1$) has CRS for all x and has marginal products that are independent of the scale;
- A technology exhibits decreasing returns to scale iff $f(t\mathbf{x}) < tf(\mathbf{x})$, for all $t>1$;
- A technology exhibits increasing returns to scale iff $f(t\mathbf{x}) > tf(\mathbf{x})$, for all $t>1$.

Example: Cobb-Douglas, V, p.12 and 14.

2. Profit maximization

2.1 Profit maximization problem

The profit maximization problem for a firm facing given prices is to choose “netputs” to maximize profits. The maximum profit that can be obtained with a technologically feasible production plan for a given price vector **p** is given by the *profit function* Π. The firm thus solves

$$\Pi(p) = \underset{y}{Max}\, py \ s.t. \ y \in Y$$

or, using the transformation function,

$$\Pi(p) = \underset{y}{Max}\, py \ s.t. \ T(y) = 0 \,.$$

The FOC’s for profit maximization imply

$$\frac{p_i}{pj} = \frac{\frac{\partial T}{\partial x_i}}{\frac{\partial T}{\partial xj}}, i, j = 1, \ldots, n-1 \,.$$

The ratios of the partial derivatives of *T* describe the technological relationship between two variables. The interpretation depends on whether the goods in question are inputs or outputs: (a) if both netputs are inputs this corresponds to the technical rate of substitution TRS_{ij} and determines the optimal input mix; (b) if *i* is input and *j* an output this corresponds to MP_j and gives the optimal input level (c) If both netputs are outputs this corresponds to MRT_{ij} and gives the optimal output mix.

- With one output:

If the firm produces only one output, the problem can be written as

$$\Pi(p) = \underset{x}{Max}\ pf(x) - wx\ s.t.\ x \geq 0,$$

where p denotes output price and w is a vector of input prices.

The FOC's for profit maximization in the latter case are

$$\frac{\partial f(x^*)}{\partial x_i} \leq \frac{w_i}{p}, x_i^* \geq 0, (p\frac{\partial f(x^*)}{\partial x_i} - w_i)x^*_i = 0, i = 1,\ldots,n-1.$$

In an interior solution, the marginal revenue product should equal the marginal cost.

Remark: The SOC for this case requires the Hessian matrix $\mathbf{D^2f(x)}$, which is a symmetric matrix, to be negative semi-definite at the optimum, i.e., $\mathbf{hD^2f(x^*)h}^t \leq 0$, for all $\mathbf{h}$.

Example: Cobb-Douglas, V, p.30.

2.2 Implications of profit maximization

a. Demand and supply functions

The *factor demand function* x(p,**w**) gives the optimal choice of inputs for each vector of prices (p,**w**); the function y(p,**w**)=f(x(p,**w**)) is called the supply function.

b. Comparative statics

Given a list of price vectors $\mathbf{p}^t$ and the associated optimal "netput" vectors $\mathbf{y}^t$, t=1,…,T, a necessary condition for profit maximization is that $\mathbf{p}^t\,\mathbf{y}^t \geq \mathbf{p}^t\,\mathbf{y}^s$ for all t,s=1,…,T. This is the Weak Axiom of Profit Maximization (WAPM) and it implies that $\triangle\mathbf{p}\triangle\mathbf{y}\geq 0$.

3. The profit function

3.1. Properties

a. $\Pi(\cdot)$ is non-decreasing (non-increasing) in p_i if *i* is an output (input);

b. $\Pi(\cdot)$ is homogeneous of degree 1 in $\boldsymbol{p}$;

c. $\Pi(\cdot)$ is convex in $\boldsymbol{p}$; intuition: by responding to the price change and adjusting the output level or input mix, the firm is at least as well of as if it did not respond;

d. $\Pi(\cdot)$ is continuous in $\boldsymbol{p}$;

e. *Hotelling's lemma*: $\frac{\partial \Pi(p)}{\partial p_i} = y_i(p),$ for all i=1,…,n, assuming that the derivative exists and $p_i > 0$.

Proofs: V, p. 41, 43, and 44

3.2. Implications

a. a. above implies: $\frac{\partial \Pi(p)}{\partial p_i} > (<) 0$ iff *i* is an output (input);

b. b. above implies y(·) and x(·) are homogeneous of degree 0;

c. c. above has comparative statics implications, namely it implies that $\frac{\partial y_i}{\partial p_i} > 0$

   and $\frac{\partial y_i}{\partial p_j} = \frac{\partial y_j}{\partial p_i}$, for all i,j=1,…,n.

4. Cost minimization

4.1 Cost minimization problem

Assume that there is a <u>single output</u> to be produced and consider the problem of minimizing the cost of producing an output level y given a vector of input prices **w**:

$$c(y,w) = \underset{x}{Min}\, wx \ s.t. \ f(x) = y, \ x_i \geq 0, \ i = 1,...n-1 .$$

The *cost function* c(·) gives the minimum cost of producing y for input prices **w**. The FOC's imply

$$w_i \geq \lambda \frac{\partial f(x^*)}{\partial x_i}, x_i^* \geq 0, (w_i - \lambda \frac{\partial f(x^*)}{\partial x_i}) x^*_i = 0, i = 1,...,n-1$$

and $f(x^*) = y$.

In an interior solution, the TRS equals the ratio of input prices, i.e., the economic rate of substitution.

Remark: The SOC requires the so called bordered Hessian for the Lagrangian to be positive semi-definite. (This means that the determinant and the principal minors should have the same sign, which, however, alternates with the number of constraints and is positive if the number is even and negative if it is odd.)

The solutions of the above problem are the *conditional input demands*, denoted by $x_i(\boldsymbol{w}, y)$. It also follows that $\lambda^* = w_i \Big/ \frac{\partial f}{\partial x_i}(x^*)$. This tells us how much the cost increases if we tighten the constraint by requiring an extra unit of y, i.e., λ equals the marginal cost.

Example: Cobb-Douglas, V, p.54.

4.2. Implications of cost minimization: comparative statics

Given a list of input price vectors $\mathbf{w}^t$ and the associated optimal factor levels $\mathbf{x}^t$, t=1,…,T, an obvious necessary condition for cost minimization is that $\mathbf{w}^t\,\mathbf{x}^t \leq \mathbf{w}^t\,\mathbf{x}^s$ for all t,s such that $\mathbf{y}^s \geq \mathbf{y}^t$. This is the Weak Axiom of Cost Minimization (WACM) and implies that $\triangle\mathbf{w}\triangle\mathbf{x} \leq 0$.