On recovery and intensity's correlation: a new class of credit risk models

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Empirical literature increasingly supports that both the probability of default (PD) and the loss given default (LGD) are correlated and driven by macroeconomic variables. Paradoxically, there has been very little effort from the theoretical literature to develop credit risk models that would include this possibility. The goals of this paper are, first, to develop the theoretical, reduced-form framework needed to handle stochastic correlation of recovery and intensity, proposing a new class of models; second, to understand under what conditions would our class of models reflect empirically observed features; and, finally, to use a concrete model from our class to study the impact of this correlation on credit risk term structures. We show that, in our class of models, it is possible to model directly empirically observed features. For instance, we can define default intensity and losses given default to be higher during economic depression periods – the well-known credit risk business cycle effect*. Using the concrete model, we show that in reduced-form models different assumptions (concerning default intensities, distribution of losses given default and specifically their correlation) have a significant impact on the shape of credit spread term structures and consequently on pricing of credit products as well as credit risk assessment in general. Finally, we propose a way to calibrate this class of models to market data and illustrate the technique using our concrete example using US market data on corporate yields.*

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1 INTRODUCTION

Recent empirical studies show that there is a significant systematic risk component in defaultable credit spreads: see Duffee (1998); Jarrow and Turnbull (2000); Duellmann and Trapp (2000); Frye (2000a, 2003); Elton *et al* (2001); Bongini *et al* (2002); Xie *et al* (2004); Elton and Gruber (2004); Altman *et al* (2005); or Chen (2007). The model underlying the Basel II internal ratings-based capital calculation (see Basel Committee (2003) and Wilde (2001)) measures credit portfolio losses only, that is, portfolio losses that are due to external influences and hence cannot be diversified away. This gives us an indication of what the main concerns are in practice and highlights the need for a realistic model of *systematic risk*. Moreover, both the probability of default (PD) and the loss given default (LGD) are key in accessing expected capital losses and measuring the exposure of portfolios of defaultable instruments to credit risk. It is, therefore, important not to ignore the interdependence between PD and LGD, since ignoring the interdependence could lead to underestimation of the true risk borne by portfolio holders. In fact, there has been increasing support, on the empirical literature, agreeing on two observed facts: (i) PD and LGD are correlated, and (ii) macroeconomic risks are likely to affect both these variables. (See Frye (2000b); Hu and Perrandin (2002); Allen and Saunders (2003); Altman *et al* (2005); or Giese (2005).) Nonetheless, most of the theoretical literature considers models in which only the default intensity, or equivalently the PD, is dependent on a state variable, assuming that the LGD is either fixed or at least independent of default intensities: see JP Morgan (1997); Wilson (1997); Saunders (1999); Gordy (2000); or Schönbucher (2001).

The purpose of this study is to present, based on the theoretical flexibility of *doubly stochastic marked point processes* (DSMPP), a reduced-form multiple default family of models that considers the influence of macroeconomic risks on PD *and* LGD.

We start by developing a theoretical family of reduced-form models that is consistent with correlated PD and LGD. Theoretically, such a correlation results from considering that having both PD and LGD dependent upon a common vector[1] stochastic state variable X. Since X affects both the jump probability and jump size, we must use DSMPP. To the best of our knowledge, this is the first time DSMPP is used in credit risk. DSMPP include as special cases all previously studied processes, including (the most used) Cox processes.

At a second stage, we investigate the consequences, within our class of models, of taking into account the empirically observed facts about PD, LGD and their correlation. We show that models that are consistent with empirically observed facts, what we call *realistic models*, should, additionally, satisfy six functional properties. Given the systematic nature of the empirical facts we deal with, we use, at this stage, a *market index* as a proxy for macroeconomic conditions. One of the advantages of

[1] Since X is multidimensional, it can, in principle, include various macroeconomic variables and firm-specific variables. The theoretical results hold for a generic X.

using such a proxy is that we can easily allow the local volatility of the index to depend negatively on its level. It is well known that market uncertainty and its level are negatively correlated.[2]

To prove relevance of the new proposed class of credit risk models, we must rely on numerical simulations, as models satisfying our realistic properties are not likely to be tractable. At this third and final stage of the paper, we aim to show that different reduced-form models' assumptions (concerning default intensities, distribution of losses given default, especially their correlation) have a significant impact on the shape of credit spread term structures and consequently on pricing of credit products and credit risk assessment in general. Thus, if empirical evidence tells us that PD and LGD are correlated, we should use a class of models that allow for these features since not using such a class might lead to biased assessment of credit risk. We use a simple model of our class to simulate. Surprisingly enough, even in the context of our simple model, correlation between PD and LGD seems to be able to capture some empirically observed properties of the term structure of credit spreads. Clearly, a more sophisticated model could have been suggested, but, as it turns out, this simple model suffices to prove the relevance of its class.

The main contributions of this study can be summarized as follows. (1) We suggest a multiple default reduced-form model in which we use the flexibility of DSMPP to model the influence of systematic risks on both PD and LGD. (2) We identify model properties that are consistent with empirically observed qualitative relationships between macroeconomic conditions and intensity of default, recovery given default or credit spreads. We claim that these properties should hold in realistic theoretical models. (3) Using a concrete model, we quantify results and simulate realistic behaviors of the term structure of credit spreads, showing that the correlation between PD and LGD (resulting from the influence of the systematic risk) must be considered. (4) Finally, we show how to calibrate our model to market data.

The rest of the paper is organized as follows. In Section 2, we set up the framework and summarize the main theoretic results concerning the use of DSMPP in credit risk models. In Section 3, we propose a family of macroeconomic models, presenting the index dynamics and justifying the assumptions about the influence of such risks on the intensity and recovery processes using empirical facts. We derive qualitative results on the influence of the market index on default intensity, recovery and credit spreads. In Section 4, we present a concrete instance of our class of models, simulating it to assess the impacts of our qualitative assumptions in terms of credit spread term structures. We end the section discussing the calibration issues of models with no closed-form solution and calibrate our concrete model to US market data. Section 5 concludes the paper, summarizing the main results and suggesting directions for future research.

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[2] Periods of recession (low index level) also tend to be periods of high uncertainty (high index volatility), reflecting some sort of market panic, while periods of economic boom are perceived as safe periods with low uncertainty.

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2 SETUP AND THEORETIC RESULTS

We consider a financial market living on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{Q}, (\mathcal{F}_t)_{0 \le t \le T})$, where $\mathbb{Q}$ is the risk-neutral probability measure. The probability space carries a multidimensional Wiener process, W, and, in addition, a DSMPP, $\mu(\mathrm{d}t, \mathrm{d}q)$, on a measurable mark space $(E, \mathcal{E})$ to model the default events. The filtration $(\mathcal{F}_t)_{0 \le t \le T}$ is generated by W and μ: $\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_t^\mu$. We assume the existence of a liquid market for default-free zero-coupon bonds for every possible maturity T. We denote the price at time t of a default-free zero-coupon bond with maturity T by $p(t, T)$. The instantaneous forward rate with maturity T is denoted $f(t, T)$, and the default-free short rate is denoted $r(t) = f(t, t)$.

In addition to the risk-free bond market, we consider a defaultable bond market. We assume that each company on the market issues a continuum of bonds with maturities T, and we allow for, possibly, multiple defaults. Even though the multiple default assumption is not the most common in credit models, it is popular for two reasons: first, it represents a better defaults procedure; and, second, at least in the classic Cox process setup, considering a multiple default model is equivalent to assuming a recovery of market value in a single default model.[3] In this paper, we choose it for the first reason. Indeed, various events are recognized as default events, besides bankruptcy. In fact, bankruptcy is only one of the events recognized as a credit event by the International Swaps and Derivatives Association (ISDA),[4] the others being failure to pay (on one or more obligations), restructuring (including reduction of interest and/or principal, postponement of payment of interest and/or principal, change of currency, contractual subordination), rating downgrades, repudiatium and moratorium. In all other events, a firm continues to operate despite the fact that its assets will be worth less. This can, directly or indirectly, be interpreted as a decrease in the face value of outstanding debt and hence be in accordance with the standard multiple default assumption. Debt holders are likely to accept the renegotiation of their claims (accepting to lose some fraction q of the face value of the claims) in order to avoid bankruptcy, which is typically costly. Schönbucher (2003) also defends multiple default models as realistic as they mimic the effect of a rescue plan as it is described in many bankruptcy codes. Finally, if we would like DSMPP to be used in top-down models of portfolio credit risk, the multiple default setup is necessary.[5]

Next we introduce some notation concerning the defaultable bond market. Assumptions 2.2 and 2.3 then characterize the default events and the dependence of

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[3] See Schönbucher (2003) 135–138, for further details on the Cox process case. An interesting question for future research is to check that this also holds in the more general DSMPP case. Our conjecture is that it does.

[4] The ISDA officially defines what can be considered a financial event related to a legal entity that triggers specific protection provided by any credit derivative contract.

[5] Top-down models of portfolio credit risk directly model the loss process of a porfolio of credits. Such loss process, naturally, must allow for several defaults. For further details, we refer the reader to Gieseke (2008).

both the default intensity and the recovery rate distribution on an abstract stochastic state variable X.

DEFINITION 2.1 **(Basic definitions)**

- The *loss quota*, q, is the fraction by which the promised final payoff of the defaultable claim is reduced at each time of default.
- The *remaining value*, after all reductions in the face value of the defaultable claim due to defaults in the time interval $[0, t]$, is denoted $V(t)$.
- At time t, the *price of a defaultable zero-coupon bond* with maturity T and the face value 1 is $\bar{p}(t, T)$. The payoff at time T of the bond is, thus, $V(T)$ ie, $\bar{p}(T, T) = V(T)$.
- We define the *instantaneous defaultable forward rate*, $\bar{f}(t, T)$, similarly to its risk-free equivalent: $\bar{p}(t, T) = V(t)\exp\{-\int_t^T \bar{f}(t, s)\mathrm{d}s\}$ and $\bar{f}(t, T) = -\dfrac{\partial}{\partial T}\ln \bar{p}(t, T)$.
- The *defaultable short rate* is defined as $\bar{r}(t) = \bar{f}(t, t)$.
- The *short credit spread*, $s(t)$, is defined as the difference between the defaultable and non-defaultable short rates $s(t) = \bar{r}(t) - r(t)$.
- The *forward credit spread* $s(t, T)$ is defined as the difference between the defaultable short rate and non-defaultable forward rates, $s(t, T) = \bar{f}(t, T) - f(t, T)$.

ASSUMPTION 2.2 **(State variable)** *There exist an underlying stochastic state variable X whose dynamics, under the risk-neutral measure $\mathbb{Q}$, is given by:*

$$\mathrm{d}X_t = \alpha_X(t, X_t)\mathrm{d}t + \sigma_X(t, X_t)\mathrm{d}W_t \tag{1}$$

where X is possibly multidimensional and α_X and σ_X are adapted processes.

ASSUMPTION 2.3 **(Default events)** *We assume the following:*

1) *Default happens at a sequence of the stopping times $\tau_1 < \tau_2 < \cdots$, where τ_i is the time of the ith jump of a point process, μ.*
2) *At each default time τ_i, the loss quota q_i is drawn from the mark space $E = (0, 1)$.*
3) *There is no total loss at default, ie, the loss quota $q_i < 1$ for all $i = 1, 2, \ldots$.*
4) *Both the arrivals of default times $(\tau_i)_{i\geq 1}$ and the distribution of the loss quotas given default $(q_i)_{i\geq 1}$ depend upon the stochastic state process X.*

Given that at each default time τ_i, the final claim amount is reduced by a loss quota q_i to $(1 - q_i)$ times what it was before, we obtain:

$$V(t) = \prod_{\tau_i \leq t}(1 - q_i) \tag{2}$$

where q_i is the stochastic marker to the default time τ_i. Obviously, in case of no default on the interval $[0, T]$, $V(t) = 1$.

Before we formally define DSMPP, we clarify the filtrations notation.

Definition 2.4 **(Filtrations)** The filtration generated by $W(t)$, $(\mathcal{F}_t^W)_{t\geq 0}$, is the background filtration.[6] The filtration $\mathcal{G}^W = \bigvee_{t\geq 0} \mathcal{F}_t^W$ contains all future and past information on the background process W. The full filtration results from combining $(\mathcal{F}_t^W)_{t\geq 0}$ and the filtration $(\mathcal{F}_t^\mu)_{t\geq 0}$, generated by the marked point process (MPP) μ, $\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_t^\mu$. Finally, $\mathcal{G}_t^W = \mathcal{G}^W \vee \mathcal{F}_t^\mu$ is the filtration generated by *all* the information concerning the background process W and only past information on the MPP μ.

Although we consider X to be Wiener driven, it is straightforward to generalize its dynamics to general jump-diffusion processes. Unfortunately, what is not possible is to make X dependent on the default events themselves, as we need the default process μ, given the information generated by the state variable, to be a true MPP. A modeling consequence of this fact is that we cannot take into account possible feedback effects from default events into X.[7] This is a drawback of the reduced-form approach to credit risk, not of our class of models in particular. Still, at least when we take the single-firm perspective, it is reasonable to suppose that its default may be sensitive to macroeconomic conditions but not the reverse. In the multi-firm setup, caution in the interpretation of results may be required.

Definition 2.5 **(DSMPP)** We call the marked point process $\hat{\mu}$ an $\mathcal{F}_t^\mu$- *MPP* if there exists a deterministic measure $\hat{\nu}$ on $\mathbb{R}_+ \times E$ such that:

$$\mathbb{P}\left(\hat{\mu}((s,t] \times B) = k \,\middle|\, \mathcal{F}_s^\mu\right) = \frac{(\hat{\nu}((s,t] \times B))^k}{k!}\, e^{-\hat{\nu}((s,t]\times B)}, \quad \text{a.s.,} \quad B \in E$$

We call the Marked Point Process μ an $\mathcal{G}_t^W$- *DSMPP* if there exists a $\mathcal{G}^W$-measurable random measure ν on $\mathbb{R}_+ \times E$ such that:

$$\mathbb{P}\left(\mu((s,t] \times B) = k \,\middle|\, \mathcal{G}_s^W\right) = \frac{(\nu((s,t] \times B))^k}{k!}\, e^{-\nu((s,t]\times B)}, \quad \textit{a.s.,} \quad B \in E$$

We note that the previous literature on credit risk have only used Cox processes (also known as doubly stochastic Poisson process).[8] The focus has been on modeling the jump intensity λ, and recovery has been assumed independent of λ. In factor models,

[6] In our setup, all the default-free processes are adapted to $\left(\mathcal{F}_t^W\right)_{t\geq 0}$.

[7] In particular, given the credit crisis of 2007, one could argue that the default risk in an economy can have important feedback effects on its macroeconomy.

[8] Recall that a counting process $N = (T_n)$ adapted to right-continuous filtration is a $\mathcal{G}_t^W$-Cox process if there is a $\mathcal{G}^W$-measurable random measure ν satisfying $\mathbb{P}\left(N(s,t] = k \,\middle|\, \mathcal{G}_s^W\right) = \frac{(\nu((s,t]))^k}{k!}\, e^{-\nu((s,t])}$, a.s $k \in \mathbb{N}$.

the state variable would influence only λ. As far as we know, this study is the first using doubly stochastic *marked* point processes in credit risk. Our goal is to allow both intensity and recovery to be affected by our state variable X. We, thus, need to model default events using a $\mathcal{G}_t^W$-DSMPP whose compensator depends on the stochastic state variable X presented in (1). Important results for what follows relate to the existence and construction of such processes. We start by showing the existence of DSMPP and then suggest a practical way to construct its compensator. Due to its rather technical level, we present the proof of the next theorem in the appendix.

THEOREM 2.6 **(Existence)** *Assume that a random measure ν on $\mathbb{R}_+ \times E$ admits the representation $M_t(\mathrm{d}q, X_t)\mathrm{d}t$, where $M_t(\mathrm{d}q, x)$ is a deterministic measure on E for any fixed x and t. Let $\hat{\nu}(\mathrm{d}t, \mathrm{d}q) = m_t(\mathrm{d}q)\mathrm{d}t$ be a deterministic compensator for some MPP $\hat{\mu}$. Assume that:*

1) *$M_t(\mathrm{d}q, x)$ is measurable with respect to $\mathcal{G}^W$; and*
2) *$M_t(\mathrm{d}q, x)$ is absolutely continuous with respect to $m(t, \mathrm{d}q)$ on $\mathcal{E}$, ie, $M_t(\mathrm{d}q, x) << m_t(\mathrm{d}q)$.*

Then, there exists a $\mathcal{G}_t^W$-DSMPP μ, such that its compensator is of the form:

$$\nu(\mathrm{d}t, \mathrm{d}q, \omega) = \nu(\mathrm{d}t, \mathrm{d}q, X_t) = M_t(\mathrm{d}q, X_t)\mathrm{d}t, \qquad \mathbb{Q} - a.s. \tag{3}$$

Given the existence of DSMPP, we can now focus on the compensator's construction. Good credit risk models consistently price credit products that depend only on default intensity (eg, digital swaps), on recovery given default (eg, recovery swaps) and on both (eg, credit default swaps). This means that we would like to model the intensity of default $\lambda(t, X_t)$ and the instantaneous conditional loss quota distribution $K(t, \mathrm{d}q, X_t)$, separately allowing both these quantities to depend on the state variable X, and not to model the whole compensator $\nu(\mathrm{d}t, \mathrm{d}q, X_t)$ to start with. Luckily, since $M_t(\mathrm{d}q, X_t)$, the multiplication of the two quantities $K(t, \mathrm{d}q, X_t)$ and $\lambda(t, X_t)$, is a random measure on $\mathbb{R}_+ \times E$, we are always allowed to do so (see, for example, Last and Brandt (1995) for further details). It, then, follows directly from Theorem 2.6 that there will always exist a $\mathcal{G}_t^W$-DSMPP whose compensator is given by $\nu(\mathrm{d}t, \mathrm{d}q, X_t) = K(t, \mathrm{d}q, X_t)\lambda(t, X_t)\mathrm{d}t$. Given this, we propose the following construction procedure.

REMARK 2.7 **(Construction procedure)** We construct the DSMPP μ as follows:

1) We specify the Wiener-driven stochastic state variable X.
2) We specify the intensity $\lambda(t, X_t)$ as a function of the state variable.
3) We specify the instantaneous conditional loss quota distribution as a function of the state variable $K(t, \mathrm{d}q, X_t)$.[9]

[9] In all the practical applications, we suppose that the instantaneous conditional loss quota distributions $K(t, \mathrm{d}q, X_t)$ are absolutely continuous with respect to the Lebesgue measure on $\mathcal{E}$, that is, we consider conditional loss quota distributions of the form $K(t, \mathrm{d}q, X_t) = \tilde{K}(t, q, X_t)\mathrm{d}q$, thus $M_t(\mathrm{d}q, x) << \mathrm{d}q$. This is enough to cover all the most common conditional distributions.

4) Finally, we construct the stochastic compensator $\nu(\mathrm{d}t, \mathrm{d}q, X_t) = K(t, \mathrm{d}q, X_t)$ $\lambda(t, X_t)\mathrm{d}t$.

We conclude this section stating some general results on credit spreads and commenting on the choice for modeling under the risk-neutral measure $\mathbb{Q}$ and its consequences in terms of the objective probability measure $\mathbb{P}$. The proof of the next proposition can be found in the appendix.

PROPOSITION 2.8 *Given Assumption 2.3 and under the martingale measure $\mathbb{Q}$:*

1) *The short credit spread, $s(t)$, is always positive, and its functional form is given by $s(t) = \lambda(t, X_t)q^e(t, X_t)$, where $q^e(t, X_t) = \int_0^1 qK(t, \mathrm{d}q, X_t)$ can be interpreted as the locally expected loss quota (which is positive for $q > 0$).*
2) *The forward credit spread $s(t, T)$ takes the form:*

$$s(t,T) = \frac{\mathbb{E}_t^{\mathbb{Q}}[\{r(T) + \lambda(T, X_T)q^e(T, X_T)\}\, e^{-\int_t^T \{r(s)+\lambda(s,X_s)q^e(s,X_s)\}\mathrm{d}s}]}{\mathbb{E}_t^{\mathbb{Q}}[e^{-\int_t^T \{r(s)+\lambda(s,X_s)q^e(s,X_s)\}\mathrm{d}s}]} - f(t,T) \tag{4}$$

3) *Defaultable bond prices can be written as $\bar{p}(t, T) = V(t)\mathbb{E}_t^{\mathbb{Q}}[e^{-\int_t^T \bar{r}_s \mathrm{d}s}]$.*

REMARK 2.9 **($\mathbb{P}$ considerations)** Our setup has been defined under the martingale measure $\mathbb{Q}$. It is possible to show that if the market price of jump risk ϕ is a deterministic function of time then:[10]

1) The $\mathbb{Q}$ and $\mathbb{P}$ default intensities relate to each other by $\lambda^P(t, X) = \lambda(t, X)$ $(1 + \phi(t))$.
2) The $\mathbb{Q}$-loss quota distribution, conditional on default, $K_t(\mathrm{d}q, X)$, equals the conditional on default loss quota distribution under $\mathbb{P}$, $K_t^P(\mathrm{d}q, X)$.

We see that, in this case, the conditional distribution of the loss quota remains unchanged under both measures, while the objective intensity can be recovered from the risk-neutral one, simply multiplying it by a deterministic function of time. Therefore, assuming that the market price of jump risk is deterministic allows us to use empirically observed facts, in defining the properties that *realistic models* should satisfy.

3 THE MACROECONOMIC RISKS

In this section, we discuss the properties we believe *realistic credit risk models* should satisfy. All properties (represented by roman numerals throughout Section 3) result from empirically observed facts (thus, the use of the adjective "realistic") and

[10] This result is a direct application of the Girsanov Theorem. Upon request, the authors will gladly provide the proof.

are justified by economic arguments.[11] Since the empirical facts under analysis have macroeconomic nature, we interpret these properties as *systematic* properties. It is widely accepted that the correlation between PD and LGD is mainly a result of the influence on both variables of *systematic* factors: see, for instance, Duffee (1998); Jarrow and Turnbull (2000); Elton *et al* (2001); Bongini *et al* (2002); Frye (2003); Allen and Saunders (2003); Xie *et al* (2004); Altman *et al* (2005); or Chen (2007). The main economic reasoning being that in recessions it is reasonable to expect more defaults (so higher PD) and thus negative relationship between PD and market conditions. Moreover, when the entire market is down, the market value of any firm's assets should be lower and debt holders should recover less if a default occurs (so higher LGD), again resulting in a negative relationship between LGD and market conditions.

In this section we use a *market index* as a proxy for macroeconomic conditions. At this stage we decided to reduce the dimension of our general state variable X to one, which we will denote as I. The main reasons are (1) most empirical studies our properties relate to rely on single variables or market indexes, (2) we can better understand the implications of the empirically observed facts in terms of model properties (how intensity or loss quota distributions should be modeled), and (3) it also allows us to include additional realism in the dynamics of the index itself. Some authors take stock market index as a proxy for macroeconomic conditions (for instance, see Gatfaoui (2006)); others prefer to take prices of important traded assets (eg, oil prices). Finally, one can also argue that there are macroeconomic indexes that aggregate the interaction between multiple sources of macroeconomic factors, such as foreign exchange rates, inflation rates, gross domestic product, economic sentiment, etc, in one single number. Examples of such indexes are the Stock and Watson Index and its successor the Chicago Fed National Activity Index (CFNAI).[12] Our *market index*, I, can be any of these indexes or traded assets as, at all times, we consider both the case when this is the price of a traded asset and the case when it is not the price of any traded asset.

We start by modeling the dynamics of our market index I. Market uncertainty and its level are negatively correlated. Periods of recession (low index level) also tend to be periods of high uncertainty (high index volatility), reflecting some sort of market panic, while periods of economic boom are perceived as safe periods and with low uncertainty. For example, Jiang and Sluis (1995) show that S&P500 has stochastic volatility. Gaspar (2001) does a comparative study of American and European stock markets (using the S&P500 and EuroStoxx 50 indexes) and shows that this feature persists across markets. Selcuk (2005) shows that the innovations to a stock market index and innovations to volatility are negatively related, especially in emerging

[11] The same properties (qualitative relations) hold under $\mathbb{P}$ and $\mathbb{Q}$ as long as the market price of jump risk is greater than -1. Recall Remark 2.9.

[12] For further information on these indexes we refer to the Chicago FED Web site: *www.chicagofed.org/economic_research_and_data/cfnai.cfm*

markets. To account for this fact, Property (i), we consider a local volatility model where index volatility is dependent on the index level.

ASSUMPTION 3.1 **(Market index)** Under the martingale measure $\mathbb{Q}$, the market index, I, satisfies the following stochastic differential equation (SDE):

$$\mathrm{d}I_t = \zeta(t) I_t \mathrm{d}t + \gamma(t, I_t) I_t \mathrm{d}W(t) \tag{5}$$

where γ is a row vector, W is a $\mathbb{Q}$-Wiener process, and we assume that I is not a price of a traded asset. If I is a price of a traded asset, we replace ζ by the short rate r.

Furthermore, for each entry γ_i, the following holds:

$$\frac{\partial \gamma_i}{\partial I}(t, I) < 0 \tag{i}$$

It is also reasonable to assume that firms may have different sensitivities to the market index. We, thus, introduce a measure of sensitivity to systematic risk, ϵ, $\epsilon \in [0, 1]$. We start by noting that if the firm's financial situation is strong enough, it should not really matter if the economy is booming or if it is in recession. That is, firms that are financially solid should be much less (or not at all) sensitive to business cycles than those in a less solid financial position. From this point of view, the parameter ϵ can also be regarded as a measure of a firm's credit worthiness. Crouchy and Mark (2001), Zhou (2001), Barnhill and Maxwell (2002) and Landschoot (2004) show that firms with a lower rating are more affected by financial and macroeconomic news than firms with a higher rating. If this is so, then it makes sense to include Properties (ii) and (iii). Property (ii) tells us that the default probability of some firms may be independent of the market situation. Property (iii) says that firms that are more sensitive to the market (have lower credit worthiness) have higher PD.

The last property of default intensities we consider, Property (iv), simply states the well-established fact that if companies are sensitive to business cycles, then there is higher PD during the recession periods (low I) than in booms (high I).

ASSUMPTION 3.2 **(The default intensity)** The intensity is a deterministic function of (t, I, ϵ). Furthermore, we have ($\bar{\lambda} \in \mathbb{R}_+$):

$$\lambda(t, I, 0) = \bar{\lambda} \tag{ii}$$

$$\frac{\partial \lambda(t, I, \epsilon)}{\partial \epsilon} > 0 \tag{iii}$$

$$\frac{\partial \lambda(t, I, \epsilon)}{\partial I} < 0 \tag{iv}$$

Concerning the loss quota distribution, we state Property (v): it is *more likely* for the loss quota, q, to be below some fixed x when index values are *low* (low I). Indeed, if a firm is in distress and going to restructure its debt during a recession, its assets are worth less and hence debt holders are more likely to accept a higher loss of their debt face value. Moreover, bankruptcy costs tend to be higher in periods of recession, due to the decreased value of firm's assets, emphasizing this effect.

Formally, this effect can be taken into account using a stochastic dominance assumption above.

Assumption 3.3 **(Loss quota)** The conditional distribution of loss quota is a deterministic function of (t, I). K is a stochastic kernel from $R_+ \times R_+ \times R_+ \to [0, 1]$ for any realization of (t, I).

We denote the cumulative distribution function of loss quota conditional on default as $\tilde{K}$:

$$\tilde{K}(t, I, x) = \int_0^x K(t, I, \mathrm{d}q), \qquad \int_0^1 K(t, I, \mathrm{d}q) = 1, \quad \forall t, I$$

with the following property: $\tilde{K}(t, I_1, x) \geq \tilde{K}(t, I_2, x)$ *if* $I_1 \geq I_2, \forall x \in R$, *ie*;

$$\frac{\partial \tilde{K}(t, I, x)}{\partial I} > 0 \tag{v}$$

$\forall t$, $\tilde{K}(t, I, x)$ stochastically dominates all the conditional distributions $\underline{I} \leq I$.

Integrating by parts, differentiating with respect to I and using (v), we can translate the above distributional properties in terms of the expected loss quota q^e.

Lemma 3.4 **(Expected loss)** Given Assumption 3.3, the following holds for the expected value:

$$q^e(t, I) = \int_0^1 qK(t, I, \mathrm{d}q), \qquad \frac{\partial q^e(t, I)}{\partial I} < 0 \tag{vi}$$

An important question is, whether there exists a tractable model within the newly defined class of models that satisfy all the above mentioned properties? A "tractable" model allows for explicit (closed-form) expressions for credit spreads. Outside the class of affine or quadratic credit spread models, it is impossible to find closed-form solutions.[13] Unfortunately, one can easily conclude that no model of affine or quadratic spreads will verify all the above properties. We note that besides the above mentioned properties, K, conditional on the state variable information, must be the distribution of a random variable taking values in (0, 1) and the intensity λ must be always positive. The following conjecture summarizes the sad news for our class of *realistic models*.

[13] For further details, see discussion in Gaspar and Schmidt (2007) or Chen, Filipović and Poor (2004).

CONJECTURE 3.5 **(Tractability)** It is not possible to find a DSMPP model satisfying properties (i)–(vi) and that allows to obtain credit spread term structures in closed form.

Still, there is something one can say about the *qualitative* impact these properties have on credit spreads.

COROLLARY 3.6 **(Short spread impact)** Given the results in Proposition 2.8, Assumption 3.2 and Lemma 3.4, the short credit spread can be rewritten as $s(t, I, \epsilon) = \lambda(t, r, I, \epsilon)q^e(t, I)$. Furthermore we have $s(t, I, 0) = \bar{\lambda}q^e(t, I)$:

$$\frac{\partial s(t, I, \epsilon)}{\partial \epsilon} = \frac{\partial \lambda(t, I, \epsilon)}{\partial \epsilon} q^e(t, I) > 0 \quad \text{and}$$
$$\frac{\partial s(t, I, \epsilon)}{\partial I} = \frac{\partial \lambda(t, I, \epsilon)}{\partial I} q^e(t, I) + \lambda(t, I, \epsilon)\frac{\partial q^e(t, I)}{\partial I} < 0$$

We note that, given a concrete functional form for the intensity λ and for the loss quota distribution, the impacts on the short credit spread can be derived explicitly and quantified. Unfortunately, this is not the situation when dealing with forward credit spreads. For the forward credit spreads, $s(t, T)$, we obtain expressions in terms of expectations (see Equation (4)) that must be simulated.

4 A CONCRETE MODEL

In this section, we illustrate the importance of the theoretical results we previously derived. We use a concrete instance of our class of realistic models and aim to highlight the importance of considering the dependence between recovery and intensity of default. We do that by showing that the different model assumptions result in substantial differences when assessing credit risk. We numerically evaluate the differences in short credit spread dynamics, forward credit spread term structures and prices of defaultable bonds. We choose a simple model to illustrate our claim. Within the class of realistic DSMPP models of credit risk, we have, of course, other more sophisticated models that could have been considered. However, for the purpose of this section, the chosen simple model is sufficient to demonstrate our ideas. Moreover, parsimonious models are appealing since they allow for a better understanding of what drives the simulation results.

Surprisingly enough, as we will show, the chosen model also replicates reasonably well some empirical evidence about the term structure of credit spreads and may help us explaining why implied volatilities from stock market indexes seem to be reasonable trackers of short credit spreads. This is an interesting by-product of our analysis.

In our concrete model, we make several simplifying assumptions. We take the risk-free rate, r, as constant and abstain from the considerations about the term structure of risk-free interest rates. Although unrealistic, it is not harmful to our goal of understanding the implications that intensity and recovery assumptions have on credit *spreads*. I is assumed to be the price of a traded asset. To consider a non-traded asset, we would need further considerations on the market price of index risk. We also take all functions to be time homogeneous; the extension to non–time homogeneous functions is straightforward.

Given these simplifications and to have a completely specified model to simulate, we need to:

- establish the dependence of the volatility of the index γ, on the index level;
- provide the intensity functional form for λ, in terms of (t, I, ϵ); and
- decide on a distribution for the loss quota, q, for all possible (t, I).

We start by defining a ratio, m, which relates the current value of the index to its long-run trend value. Let us define:

$$m(I) = \frac{I - \bar{I}}{I} \tag{6}$$

where $\bar{I}$ is the long-run trend value and *a priori* given.

The ratio m measures, in relative terms, how close to (or far away from) the long run trend value parameter, I, the current value of index I is. Intuitively, it seems reasonable to make all our functions dependent on some relative value of the index, instead of its absolute value. Long-run trend value parameter $\bar{I}$ will be assumed to grow at the risk-free rate over time.[14] We note that the higher the current level of the index, the higher is m, since $\frac{\partial m}{\partial I} = \frac{\bar{I}}{I^2} > 0$. So, m can be interpreted as a rate indicating how bullish or how bearish the market may be at every point in time. In normal markets, we have $m = 0$; in bull markets, $m > 0$; in bear markets, $m < 0$. A reasonable range for $m(I)$ is the interval $[-0.3, +0.3]$.

4.1 The market index volatility γ

Based on the ratio m, we define the *volatility of index*, $\gamma(I)$, in the following way:

$$\gamma(I) = \bar{\gamma}\,(1 - m(I))^{\frac{1}{2}} \qquad \forall I, \bar{\gamma} \in \mathbb{R}_+ \tag{7}$$

Agreeing with Assumption 3.1, the higher the current value of the index, the lower is the index volatility γ, $\frac{\partial \gamma(I)}{\partial I} = \frac{1}{2}\bar{\gamma}\,[1 - m(I)]^{-\frac{1}{2}} \left(-\frac{\partial m(I)}{\partial I}\right) < 0,\ \forall I > 0$.

Figure 1 shows us two possible paths for the index process, one assuming γ to be a constant and the other where the index volatility depends on the index level as in (7).

[14] This is consistent with our simplifying assumption that we are dealing with a price of a traded asset that, under the risk-neutral measure $\mathbb{Q}$, is supposed to grow at the risk-free rate. One always needs to be sure that the long-run level is consistent with the index dynamics.

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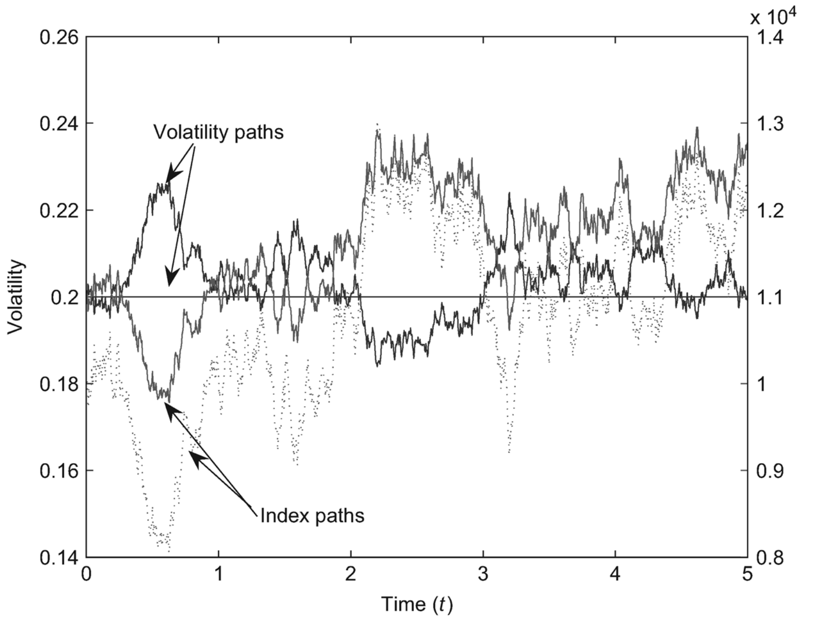
FIGURE 1 Two paths for the index level and volatility.



The same noise was used for both cases, and we took $\bar{I} = 10{,}000$ and $I = 10{,}000$. Case 1: constant volatility $\gamma = 0.2$, the index process is the full line. Case 2: stochastic volatility as in (7), the index process is the dotted line.

4.2 The default intensity λ

Having defined the index volatility, we now define the intensity function:

$$\lambda(I, \epsilon) = \bar{\lambda}\,[1 - m(I)]^{\epsilon} \quad \text{for } \bar{\lambda} \in \mathbb{R}+ \text{ and } \epsilon \in [0, 1] \tag{8}$$

We note that we can interpret the intensity function as a function of the index level or, if we prefer, as a function of the index volatility. One can argue that the intensity should not be affected by index level, but instead by its volatility since it is the volatility that represents the "risk."

The definitions in (7) and (8) allow for both interpretations as it holds $\lambda(I, \epsilon) = \bar{\lambda}\left(\frac{\gamma(I)}{\bar{\gamma}}\right)^{2\epsilon}$. As desired, the lower the index, the higher the default intensity:

$$\frac{\partial \lambda}{\partial I} = \bar{\lambda}\epsilon\,(1 - m(I))^{\epsilon - 1}\left(-\frac{\partial m(I)}{\partial I}\right) < 0 \quad \text{or}$$

$$\frac{\partial \lambda}{\partial I} = \frac{\partial \lambda}{\partial \gamma}\frac{\partial \gamma}{\partial I} = \bar{\lambda}\left(\frac{\gamma(I)}{\bar{\gamma}}\right)^{2\epsilon - 1} \quad \frac{\partial \gamma}{\partial I} < 0$$

FIGURE 2 (a) $\gamma(I)$ for different levels of $[1-m]$ versus naive constant volatility $\bar{\gamma}=0.2$. (b) $\lambda(I)$, for different levels of $m(I)$ and different $\epsilon=0, 1/16, 1/4, 1/2$, $\bar{\lambda}=0.05$.

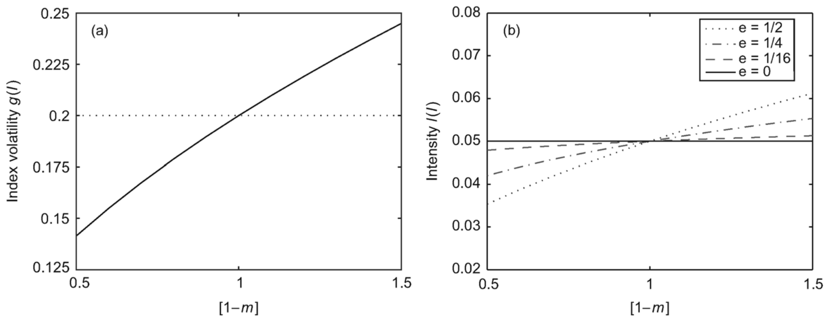


Figure 2 shows the functions $\lambda(I)$ and $\gamma(I)$ for different values of $[1-m]$.[15]

4.3 The loss quota *q*

Finally, we need to decide on the loss quota distribution. As before, we make use of the ratio m to define the dependence of loss process distribution on the market index I. We choose the beta class of distributions.[16] Concretely, we consider $q \sim Beta\,(2[1-m(I)], 2)$ (ie, we take $a = 2[1-m(I)]$ and $b = 2$), which is consistent with the desired properties referred to in Assumption 3.3. Thus:

$$\tilde{K}(q,I) = \frac{1}{B(2[1-m(I)],2)} \int_0^q x^{1-2m(I)}(1-x)\mathrm{d}x$$

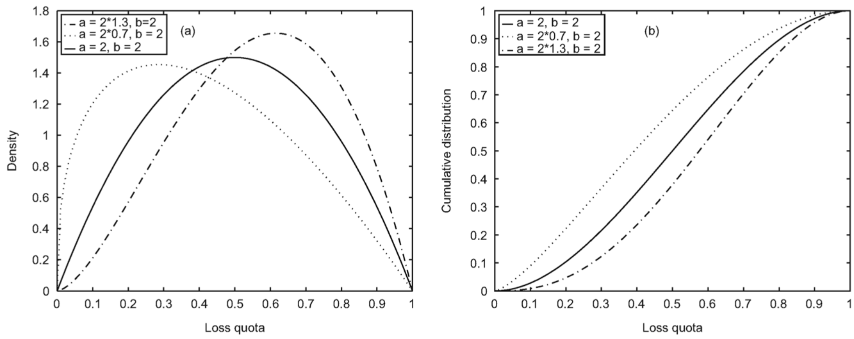
From the properties of the beta distribution, we immediately obtain the expected loss given by:

$$\begin{aligned} q^e(I) = \mathbb{E}[q(I)] &= \frac{2[1-m(I)]}{2[1-m(I)]+2} \quad \text{and} \quad \frac{\partial q^e(I)}{\partial I} = \frac{\partial q^e(I)}{\partial m}\frac{\partial m}{\partial I} \\ &= \frac{-1}{(2-m(I))^2}\frac{\partial m(I)}{\partial I} < 0. \end{aligned}$$

[15] Since in Figure 2 the x axis refers to $[1-m]$ (and not m itself), the normal market situation corresponds to a value of 1, bull market situations to values less than 1 and bear market situations to values higher than 1.

[16] Recall that the beta density function is given by $f(x) = \frac{1}{B(a,b)}x^{a-1}(1-x)^{b-1}\mathbf{1}_{(0,1)}(x)$ where $a > 0$, $b > 0$ and $B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\mathrm{d}s$ (beta function). Useful properties of the beta distribution are $\mathbb{E}[X] = \frac{a}{a+b} = \mu$, $\operatorname{Var} X = \frac{ab}{(a+b+1)(a+b)^2}$ and $\mathbb{E}[(X-\mu)^r] = \frac{B(r+a,b)}{B(a,b)}$.

FIGURE 3 (a) Density and (b) cumulative distribution functions of loss quota for $m = -0.3$, $m = 0$, $m = +0.3$.



Hence, in our concrete model if default occurs exactly at a moment when the index is at its long-run level, the expected loss quota is $q^e = \frac{1}{2}$; if default occurs instead when the index level is "high" ($m > 0$), one expects to recover more, expected loss quota decreases below $\frac{1}{2}$; if default occurs when the index level is "low" ($m < 0$), one expects to recover less, expected loss quota increases. Figure 3 shows the loss quota density and its cumulative distribution function for three different values of m.[17] As expected in good market conditions, our loss distribution is negatively skewed, as we expect to loose less, given that default occured. Figure 4(a) shows possible realizations of the loss quota (drawn from the beta density with the appropriate mean for each value of m) (stars), the expected loss quota (full line) and the naive approach of taking $\bar{q} = \frac{1}{2}$ (dotted line). Figure 4(b) shows a scatter plot of λ versus one possible recovery realization for different levels of the index.

4.4 Simulation results

In simulations, we use the Monte Carlo method with a step size equivalent to one trading day (250 steps per year) and all simulations concern 5,000 paths. The same noise matrix is used for all scenarios and cases so that the values obtained can actually be compared.[18] Table 1 reports the reference parameters, while Table 2 characterizes all possible scenarios.

[17] Note that for $m = -0.3$, we get $a = 2 \times 1.3$ and a positively skewed distribution of losses given default in bear markets. Likewise for $m = +0.3$, we obtain $a = 2 \times 0.7$ resulting in negatively skewed distribution in a bull market. For $m = 0$ (normal market), we get $a = 2$ and a symmetric distribution.

[18] As discretization errors would be in the same direction for all scenarios.

FIGURE 4 (a) Loss quota possible realizations and expected value for different values of $[1-m]$. Dotted line is the naive $q = \frac{1}{2}$. (b) Scatter plot of intensity versus a recovery realization for different values of m.

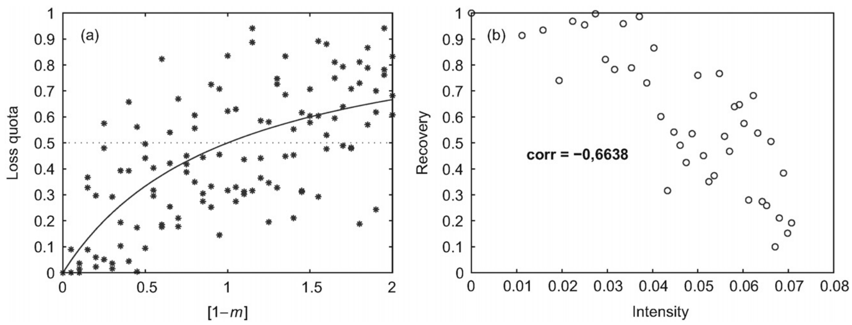


TABLE 1 Reference values for the parameters in the model.

Maturities (T)	From days up to 20 years
Risk-free interest rate	5%
$m(l)$	$\begin{cases} \text{bull market} & +0.3 \\ \text{normal market} & 0 \\ \text{bear market} & -0.3 \end{cases}$
Long-run index value	$10.000e^{0.5\times T}$
Fixed index volatility ($\bar{\gamma}$)	20%
Fixed intensity value ($\bar{\lambda}$)	5%
Fixed recovery value $\left(\bar{q} = \frac{1}{2}\right)$	50%

TABLE 2 Basic reference scenarios for simulations.

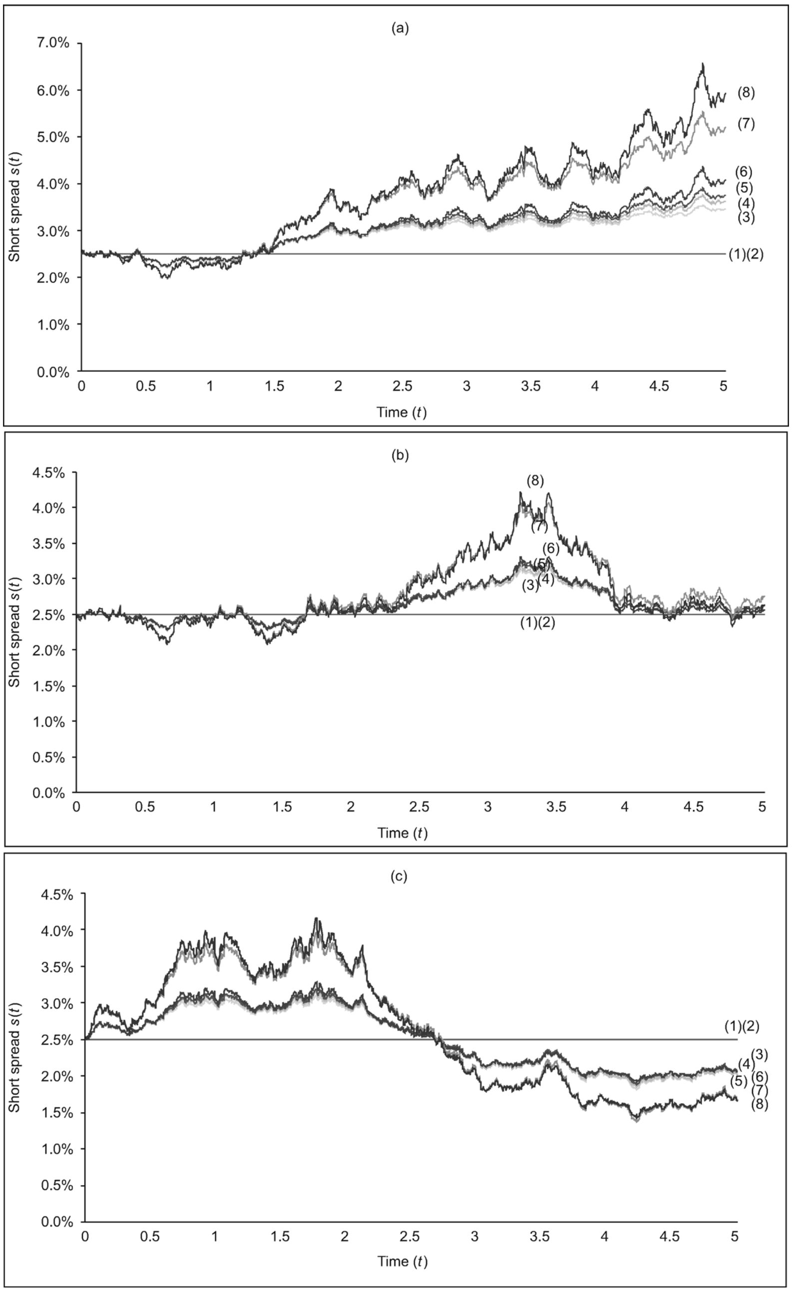
Scenario	**Index volatility**	**Intensity**	**Recovery**
(1)	F	F	F
(2)	S	F	F
(3)	F	F	S
(4)	S	F	S
(5)	F	S	F
(6)	S	S	F
(7)	F	S	S
(8)	S	S	S

F = Fixed, S = Stochastic.

4.4.1 Differences in the term structure of credit spreads

We start by looking into short spread dynamics. Figure 5 presents three possible paths for the short spread under each scenario. Obviously, three paths are not representative, still we believe the intuition is nice and we chose paths with different

FIGURE 5 Three possible paths for the short spread, $s(t)$.



(a) The market index decreases over time, leading to an increase of the short spreads;
(b) existence of mixed path; (c) the index value ends up increasing, leading to a reduction in the short spreads.

characteristics. From the analysis of this figure, we can conclude that allowing for some stochasticity, either in the intensity process or in the expected loss quota, leads to similar short spread dynamics and that it is the combined effect that really makes the difference. In any of the presented paths, if just one of the effects would be considered, the short spreads do not oscillate more than 1% below or above the naive 2.5%, while for the combined effect, the variation can be as large as 4% (in the case of path (*a*)) and quite often above 2%.

An interesting side effect of our concrete model is that when we take the index volatility to be stochastic and negatively related to the index level, the short spread dynamics can be tracked quite well by observing the index volatility. See Figure 6 with three possible volatility paths and compare with the short spread evolution in Figure 5. A natural conjecture follows.

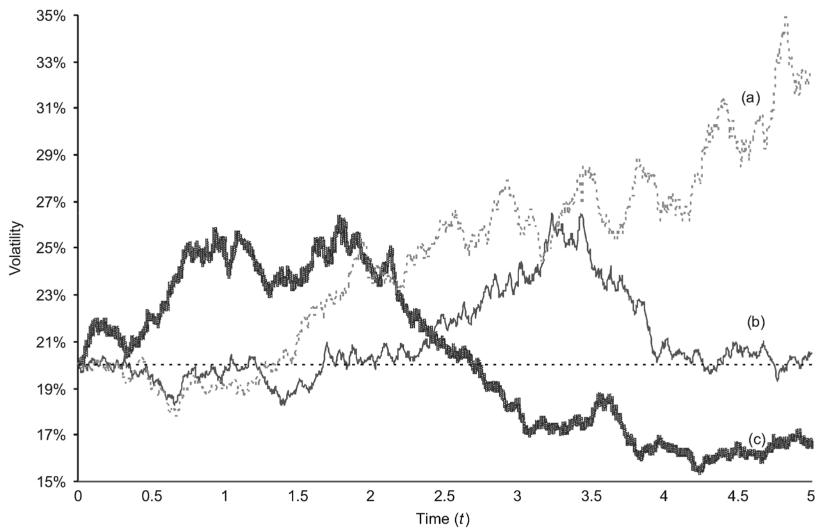
CONJECTURE 4.1 **(Implied volatility as credit spread tracker)** Since the (spot) volatility seems to be a good tracker of the short spread, then implied volatilities of options with longer maturities may be good trackers of the forward spread term structure.

This is all due to the negative correlation between the index level and its volatility. Still, it provides a fundamental reason for using implied volatilities of options on indexes as predictors of forward credit spread term structures, which seems to be common practice among traders (who use at-the-money (ATM) volatility term structures as predictors). Collin-Dufresne *et al* (2001) also investigated the determinants of credit spread and showed that credit spreads are mostly driven by a single common factor and that implied volatilities of index options contain important information for credit spreads.[19]

We now focus the analysis on the term structure of (forward) credit spreads. From Figure 7 we see that when we consider the dependence between PD and LGD and the negative relation, the index level and volatility – scenario (8), the term structure seems to converge faster to its long-run level. In fact, for maturities higher than 15 years, the term structure of this scenario is relatively flat. Thus, the forward credit spreads are most sensitive to the influence of the market index at the relatively shorter maturities, and around the 15-year maturity, the credit spreads become relatively flatter and less sensitive to the market index, moving in fact closer to each other. There is a clear lack, in the existing emprical literature, of studies about the *shape* of credit spreads term structures. Furthermore, usually, corporate debt tends to be of relatively lower matuirties than government debt, making it hard to accurately estimate credit spreads with maturities higher than 10 years. For both these reasons,

[19] Recent papers (see, for example, Cremers, Dreissen and Weinbaum (2004)) start using measures of volatility and skewness based also on individual stock options to explain credit spreads on corporate bonds. Implied volatilities of individual options are shown to contain important information for credit spreads. They showed that those implied volatilities improve on both implied volatilities of index options. However, in the suggested framework, we cannot model this feature since the reduced models do not allow us to model stock and corporate bonds together.

FIGURE 6 Volatility paths corresponding to the spot spread paths.



it is hard to check whether real-life credit spreads are flat for maturities as big as 15 years. The most relevant finding, however, is that the differences between different models are substantial at maturities liquidly traded in real life (maturities lower than 10 years) and seem to persist even in the long run. Table 3 presents spot ($T = 0$) and forward spreads (all other maturities).

4.4.2 Differences in pricing and survival probabilities

Table 4 reports defaultable (non-zero recovery) bond prices under various scenarios and market conditions for several maturities. The first point that should be highlighted is that even for low maturities, there is a difference in the prices produced by the naive scenarios (scenarios (1) and (2)), scenarios where either the PD or LGD is dependent on the index level (scenarios (3),(4),(5),(6)) and scenarios where we consider the combined effect. For the bull and bear markets, the pricing difference is clear already at $T = 0.1$. At five-year maturities the underpricing of the naive model can be up to 5% in a bull market and up to 10% in a bear market. When we consider longer horizons, from five to 15 years, in the stochastic volatility scenario, the survival probability decreases by almost 40% for the bull market and up to 50% for the bear market. In our opinion, this is a realistic feature of the model, since at the longer horizons when the market is in recession and firms are known to be sensitive to the fluctuations of the market, the PD is quite high. Moreover, from Table 5 it is interesting to note that in a bull market although a stochastic volatility scenario yields higher survival probabilities at all the maturities, the difference in

FIGURE 7 Term structure of forward spreads for all scenarios, under three possible market conditions: (a) bull market, (b) normal market and (c) bear marker.

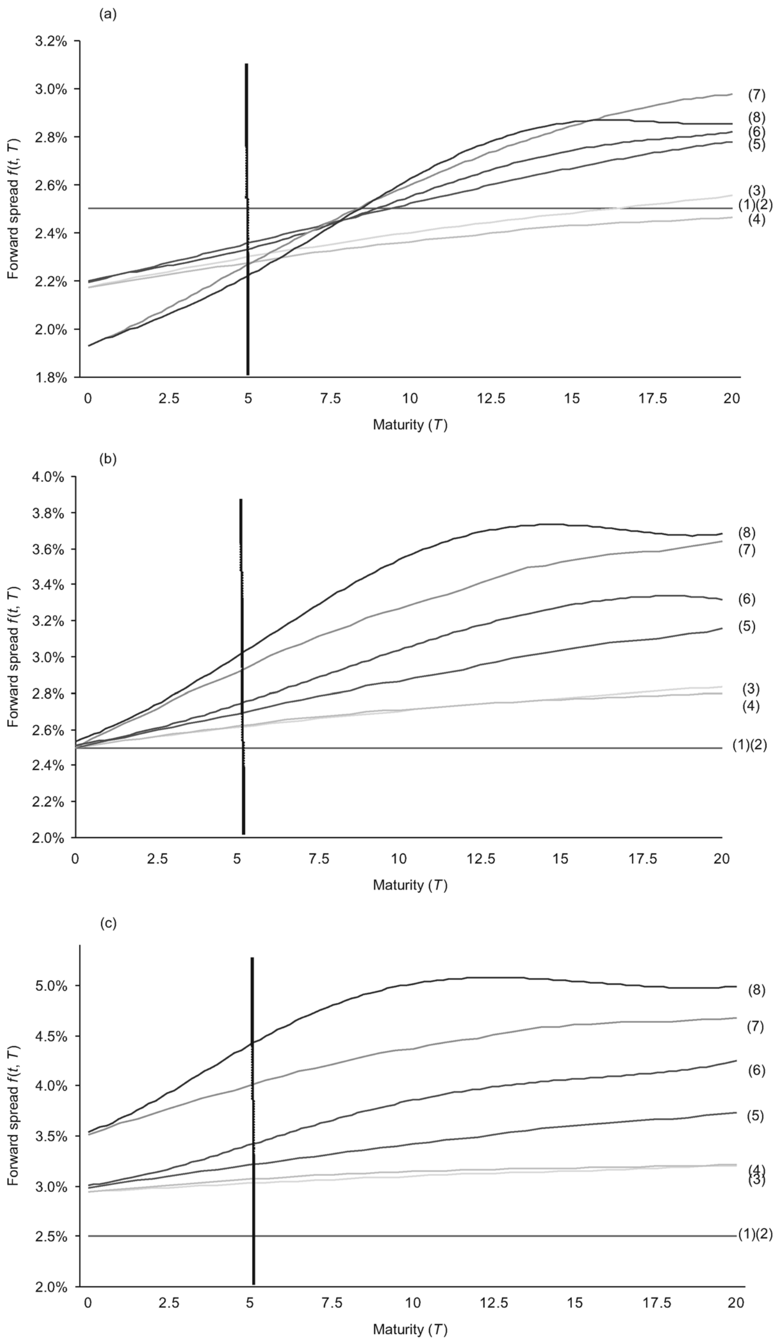


TABLE 3 Credit spreads for different scenarios and market conditions.

T	**(1)(2)**	**(3)**	**(4)**	**(5)**	**(6)**	**(7)**	**(8)**
Bull market							
0	2.5	2.174	2.174	2.192	2.192	1.906	1.906
0.5	2.5	2.188	2.185	2.209	2.206	1.945	1.937
1	2.5	2.201	2.196	2.226	2.220	1.983	1.968
2	2.5	2.228	2.219	2.262	2.251	2.063	2.040
5	2.5	2.299	2.275	2.355	2.334	2.264	2.217
7	2.5	2.343	2.317	2.425	2.414	2.412	2.383
10	2.5	2.402	2.363	2.519	2.542	2.592	2.604
15	2.5	2.482	2.430	2.674	2.769	2.855	2.898
20	2.5	2.554	2.467	2.786	2.844	2.992	2.886
Normal market							
0	2.5	2.5	2.5	2.5	2.5	2.5	2.5
0.5	2.5	2.512	2.513	2.519	2.519	2.544	2.545
1	2.5	2.525	2.525	2.538	2.539	2.587	2.589
2	2.5	2.548	2.549	2.576	2.582	2.674	2.688
5	2.5	2.612	2.615	2.684	2.718	2.908	2.977
7	2.5	2.650	2.660	2.763	2.853	3.073	3.238
10	2.5	2.701	2.705	2.867	3.024	3.264	3.493
15	2.5	2.770	2.764	3.036	3.300	3.525	3.764
20	2.5	2.835	2.799	3.155	3.383	3.643	3.709
Bear market							
0	2.5	2.941	2.941	2.988	2.988	3.516	3.516
0.5	2.5	2.951	2.955	3.010	3.020	3.567	3.590
1	2.5	2.961	2.969	3.032	3.053	3.618	3.667
2	2.5	2.980	2.994	3.074	3.117	3.712	3.808
5	2.5	3.030	3.069	3.209	3.429	4.001	4.428
7	2.5	3.058	3.107	3.298	3.612	4.174	4.735
10	2.5	3.099	3.145	3.417	3.832	4.372	4.976
15	2.5	3.151	3.181	3.605	4.110	4.607	5.089
20	2.5	3.207	3.210	3.730	4.258	4.678	4.986

Short credit spreads $T = 0$, forward credit spreads for all other maturities. All values are in percentages.

survival probabilities is much smaller at the higher maturities. In a bear market, on the other hand, survival probabilities are lower for the stochastic volatility case, and the difference in survival probabilities between the stochastic volatility and naive scenarios is more pronounced. In contrast to the bull market, the difference increases by approximately 5% when the investment horizon is extended from five to 15 years.

4.4.3 How to account for different ratings

We now take a closer look at the parameter ϵ, which we recall (Assumption 3.2) is a measure of the sensitivity of a firm's PD to the market situation. The intuition comes from the fact that the PD of firms with high credit worthiness should depend very little (or not depend at all) on the market oscillation, while less credit worthy firms are

TABLE 4 Prices of zero-coupon defaultable bonds (100 nominal value) for different scenarios and market conditions.

T	**(1)(2)**	**(3)**	**(4)**	**(5)**	**(6)**	**(7)**	**(8)**
Bull market							
0.1	99.25	99.28	99.28	99.28	99.28	99.30	99.30
0.5	96.31	96.47	96.47	96.46	96.46	96.59	96.59
1	92.77	93.06	93.06	93.04	93.04	93.28	93.29
2	86.06	86.58	86.59	86.54	86.55	86.96	86.98
5	68.72	69.63	69.66	69.49	69.53	70.13	70.21
7	59.13	60.12	60.19	59.93	59.98	60.54	60.66
10	47.22	48.19	48.29	47.89	47.92	48.32	48.43
15	32.45	33.21	33.36	32.75	32.68	32.83	32.85
20	22.31	22.81	22.99	22.26	22.15	22.09	22.17
Normal market							
0.1	99.24	99.24	99.24	99.24	99.24	99.24	99.24
0.5	96.31	96.31	96.31	96.31	96.31	96.30	96.30
1	92.77	92.75	92.75	92.75	92.75	92.72	92.72
2	86.06	86.02	86.02	86.00	85.99	85.91	85.90
5	68.72	68.52	68.52	68.40	68.36	68.00	67.92
7	59.13	58.80	58.78	58.58	58.46	57.91	57.68
10	47.21	46.70	46.68	46.33	46.05	45.31	44.83
15	32.45	31.72	31.70	31.13	30.60	29.76	29.07
20	22.30	21.47	21.48	20.77	20.20	19.38	18.82
Bear market							
0.1	99.24	99.20	99.20	99.19	99.19	99.14	99.14
0.5	96.31	96.09	96.09	96.07	96.06	95.81	95.80
1	92.76	92.34	92.34	92.29	92.28	91.78	91.76
2	86.06	85.27	85.26	85.15	85.11	84.16	84.07
5	68.72	67.06	67.00	66.70	66.44	64.52	64.00
7	59.11	57.07	56.96	56.50	55.97	53.73	52.74
10	47.20	44.78	44.63	43.97	43.05	40.67	39.18
15	32.44	29.83	29.67	28.72	27.48	25.29	23.69
20	22.30	19.82	19.69	18.62	17.41	15.61	14.38

more sensitive to business cycles. In this sense, different ϵ parameters could represent the term structure of firms with different credit ratings. In the following, we consider three different values for ϵ: high, $\epsilon = 1/2$; medium $\epsilon = 1/4$; and low, $\epsilon = 1/16$.[20] Table 6 and Figure 8 show the credit spread simulation results for the different ϵ values under normal market conditions and up to a five-year horizon. The key feature is that the term structure of less sensitive (higher ratings) firms have a flatter *slope*. This is particularly obvious for scenarios (7) and (8) when the index influences both PD and LGD and less obvious when it affects only one of them. Thus, in practice, it is important to take into account the correlation with the market index, especially when

[20] The case of total insensitivity, $\epsilon = 0$, is always considered since in scenarios (1),(2),(3) and (4), $\lambda(I, \epsilon) = \bar{\lambda}$.

TABLE 5 Survival probabilities up to time T.

	Bull market			**Normal market**			**Bear market**		
T	(1)(2)(3)(4)	(5)(7)	(6)(8)	(1)(2)(3)(4)	(5)(7)	(6)(8)	(1)(2)(3)(4)	(5)(7)	(6)(8)
0.1	99.6	99.6	99.6	99.5	99.5	99.5	99.5	99.4	99.4
0.5	97.5	97.8	97.8	97.5	97.5	97.5	97.5	97.0	97.0
1	95.1	95.7	95.7	95.1	95.1	95.1	95.1	94.1	94.1
2	90.5	91.5	91.5	90.5	90.3	90.3	90.5	88.6	88.5
5	77.9	79.6	79.7	77.9	77.2	77.1	77.9	73.4	72.9
7	70.4	72.4	72.5	70.4	69.2	68.9	70.4	64.4	63.3
10	60.6	62.5	62.6	60.6	58.5	57.9	60.6	52.8	51.0
15	47.2	48.4	48.4	47.2	43.9	42.9	47.2	37.5	35.2
20	36.8	37.2	37.3	36.8	32.6	31.7	36.8	26.6	24.4

All values are percentages.

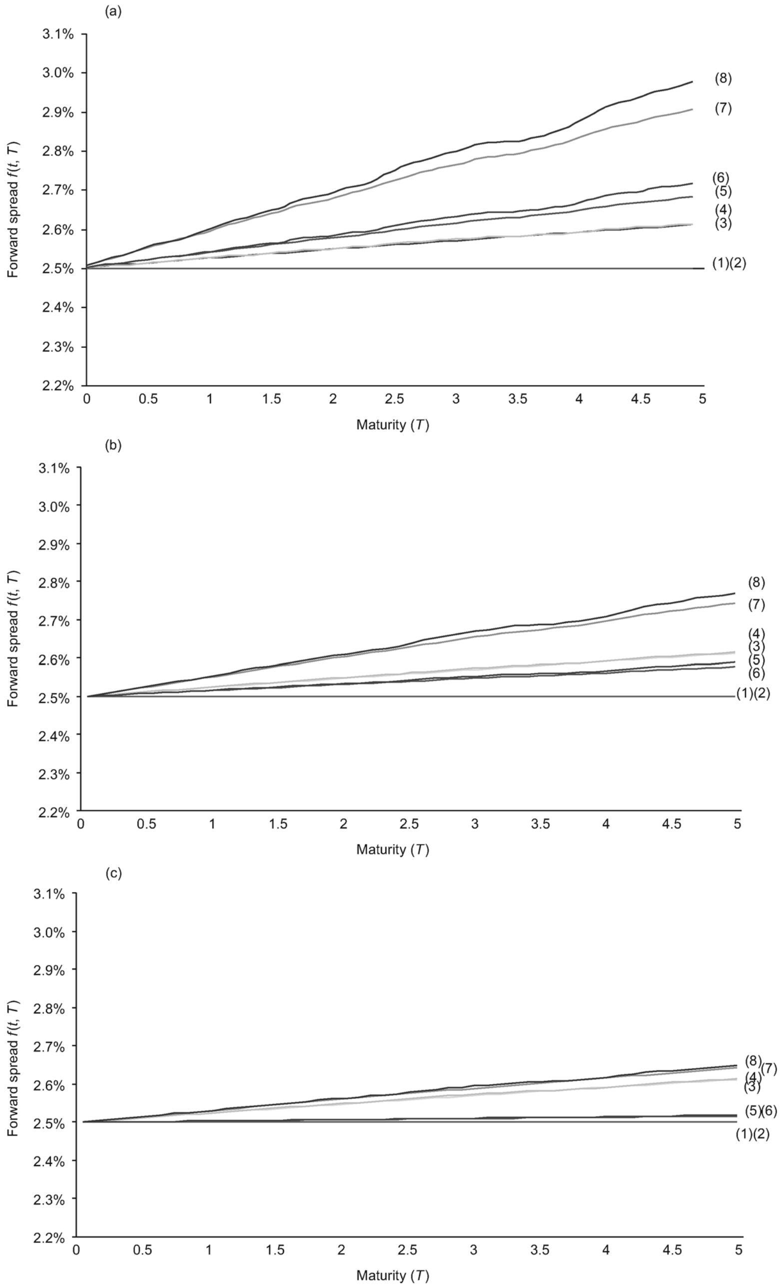
TABLE 6 Credit spreads for all scenarios and three different values of ϵ.

T	**(1)(2)**	**(3)**	**(4)**	**(5)**	**(6)**	**(7)**	**(8)**
High $(\epsilon = \frac{1}{2})$							
0	2.5	2.5	2.5	2.5	2.5	2.5	2.5
0.5	2.5	2.512	2.513	2.519	2.519	2.544	2.545
1	2.5	2.525	2.525	2.538	2.539	2.587	2.589
2	2.5	2.548	2.549	2.576	2.582	2.674	2.688
5	2.5	2.612	2.615	2.684	2.718	2.908	2.977
Medium $(\epsilon = \frac{1}{4})$							
0	2.5	2.5	2.5	2.5	2.5	2.5	2.5
0.5	2.5	2.512	2.513	2.508	2.508	2.527	2.527
1	2.5	2.525	2.525	2.516	2.516	2.553	2.554
2	2.5	2.548	2.549	2.531	2.534	2.604	2.610
5	2.5	2.612	2.615	2.577	2.588	2.743	2.770
Low $(\epsilon = \frac{1}{16})$							
0	2.5	2.5	2.5	2.5	2.5	2.5	2.5
0.5	2.5	2.512	2.513	2.502	2.502	2.516	2.516
1	2.5	2.525	2.525	2.503	2.503	2.531	2.531
2	2.5	2.548	2.549	2.507	2.507	2.561	2.563
5	2.5	2.612	2.615	2.517	2.518	2.642	2.649

All values are percentages.

considering a portfolio of securities with low credit ratings. The effect will be even more pronounced when we have stochastic index volatility. As already mentioned, the emprical literature concerning the shape of credit spreads term structure is scarce. Still, our findings seem to be in accordance with those of Krishnan *et al* (2006) and He *et al* (2007).

FIGURE 8 Term structure of forward spreads for all scenarios, under normal market conditions, and for three different values of ϵ: (a) high, $\epsilon = \frac{1}{2}$; (b) medium, $\epsilon = \frac{1}{4}$; (c) and low $\epsilon = \frac{1}{16}$.



4.5 Calibration of the model to market data

Lack of closed-form solutions complicates the calibration procedure. In this section, we present a possible solution to overcome this difficulty, using a perturbed class of models (that include our concrete model as a special case) and a first-order approximation to the yield spreads implied by that class of models. We chose this approach as credit *yields* are nothing more than averages of forward credit spreads and tend to be rather well behaved, hence, suitable to approximation methods. Our data consists of daily data from August 2004 to March 2007, on benchmark yields of Moody's Aaa and Bbb-rated long-term US corporate bonds. We use long-term US government yields as a proxy to the risk-free short rate and S&P500 as our market index.

Concretely, we specify the index volatility, intensity and the expected loss quota as follows:

$$\gamma(I) = \bar{\gamma}\,(1 - km(I))^{\frac{1}{2}}; \quad \lambda(m(I)) = \bar{\lambda}\,\{1 - km(I)\}^{\epsilon};$$
$$q^e(m(I)) = \frac{a(1 - km(I))}{a(1 - km(I)) + b}$$

Note that we still assume a beta distribution for the distribution of losses but we generalize its parameters to $a(1 - km(I))$ and b. Also if $k = 0$ the model reduces to the naive model, with constant intensity $\bar{\lambda}$ and constant expected loss quota $q^e = \frac{a}{a+b}$; if $k = 1$, it reduces to our concrete model. So, we are actually calibrating a wider class than our concrete model to market data.

Since the data comes in *yields*, we note that using the DSMPP approach, the yield spread between the defaultable and non-defaultable bonds can be computed as follows:

$$\bar{y}(t, T) - y(t, T) = -\frac{1}{T - t}\int_t^T s(t, u, k)\mathrm{d}u \tag{9}$$

with $s(t, T, k)$ as in (4) but with the extra perturbation parameter k.

In order to be able to calibrate the perturbed class of models to the observed credit yield spreads, we then do a first-order Taylor approximation of $s(t, T, k)$ around the value $k = 0$. We consider the perturbed expressions for the index volatility, intensity and expected loss quota, the m ratio definition in (6) and the market index dynamics in (5). Then, the first-order approximation to the credit yield spread as defined in (9) is given by:

$$\begin{aligned}\bar{y}(t, T) - y(t, T) &\approx -\frac{1}{T - t}\int_t^T \left\{s(t, u, 0) + \frac{\partial s(t, u, 0)}{\partial k}k\right\}\mathrm{d}u\\ &\approx \frac{\bar{\lambda}a}{a + b} + \bar{\lambda}a\frac{(1 + \epsilon)(a + b) - a}{(a + b)^2}k -\end{aligned}$$

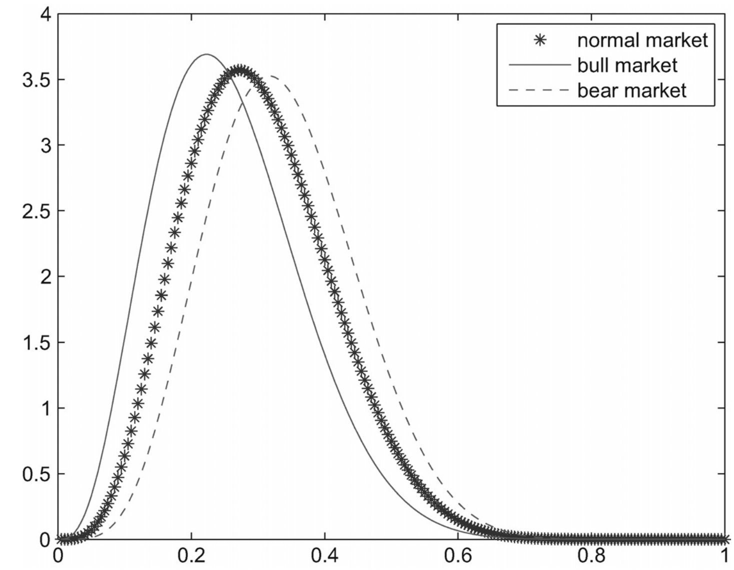
$$
\begin{aligned}
&- \frac{\bar{\lambda} a}{(a+b)^2} \frac{(1+\epsilon)(a+b)-a}{T-t} \frac{1}{\frac{\bar{\gamma}^2}{2}-r} \times \\
&\times \left\{ e^{(T-t)\left(\frac{\bar{\gamma}^2}{2}-r\right)} - 1 \right\} (1-m(I_t))\, k \qquad (10)
\end{aligned}
$$

Given (10), we compute model-consistent time series of yields for corporate bonds and find the set of parameters that minimizes the difference (in the least-square sense) to the observed term structure. Table 7 presents the estimation results. Figure 9 shows the *estimated* loss quota density for three different values of m: $m = +0.3$ representing a bull market, $m = 0$ for the case where the market is at its long run and $m = -0.3$ representing a bear market.

TABLE 7 Least-square estimates of the model parameters (standard deviations in brackets).

Default intensity $\bar{\lambda}$	0.0281	(0.0104)
Sensitivity ϵ_{Aaa}	0.0737	(1.0555)
Sensitivity ϵ_{Bbb}	0.9001	(3.7663)
Volatility of index $\bar{\gamma}$	0.35	(0.0017)
Beta-distribution parameter a	4.9698	(0.2631)
Beta-distribution parameter b	11.595	(0.2286)
Perturbation parameter k	0.6260	(0.0596)

FIGURE 9 Estimated beta distribution for loss quotas



We would like to finish this section by noting that for different data sets, other methods may be more appropriate. A recent promising bootstrapping method has been proposed by Das and Hanouna (2007). The authors use credit default swap (CDS) spreads data and apply their ideas to fixed recovery models and to the classical Merton (1974) model; their method could also be applied to our model. In fact, the only requirement is that it must be possible to write the expected loss quota as a deterministic function of the intensity and model parameters. This is clearly the case, for instance, for our generalized concrete model. We get $q^e(\lambda(t, I_t);\ a, b, \epsilon, \bar{\lambda}) = 1 - b\left[a\left(\frac{\lambda(t,I_t)}{\bar{\lambda}}\right)^{1/\epsilon} + b\right]^{-1}$.

5 CONCLUSIONS AND FUTURE RESEARCH

We introduce DSMPP to credit risk modeling and propose a class of realistic reduced-form models, in which both the PD and LGD depend on a macroeconomic index. We explain the economics behind the fact that during recessions both the PD and the LGD increase (the reverse happens during economic booms) and relate empirical evidence to functional properties of intensity and loss quota distribution. Finally, we discuss the (in)existence of tractable models that would take into account all these desirable properties.

We then use a concrete (simple) example from the class of suggested models and use simulations to compute survival probabilities, defaultable bond prices and forward spread term structures and show that allowing PD and LGD to depend on the same macroeconomic factors may help explain some empirically observed features. As a by-product of our analysis, we found that our concrete model is consistent with market (spot) volatility, tracking the short credit spreads, suggesting that the term structure of ATM-implied volatilities of index options may do the same for forward credit spreads. Given the simplicity of the proposed concrete model, we found the results to be encouraging. We, therefore, ended up the simulation section suggesting a way to calibrate, and actually calibrating (to US market data), our concrete model. Above all, during our simulations, we clearly showed that different assumptions about the PD, LGD and their correlation have significant impacts on the shape and dynamics of the credit spread term structure, survival probabilities and credit risk assessment in general.

In terms of future research, since there seems to be no hope for closed-form solutions within our class of realistic DSMPP credit models (perhaps this is the right price to pay for taking into account so many empirical observations) we suggest that new models should be analyzed where both the intensity and the distribution of the loss quota should be modeled as realistically as possible (this may involve different functional forms and/or a different market price of jump risk assumption). Also, empirical studies of the credit spreads term structure shapes observed in the market can help define such functional forms. One should also have a careful look at whether spot or implied volatilities from market indexes are indeed good trackers of credit spread term structure. Finally, another obvious direction is to extend the use of DSMPP to the modeling of credit portfolio products. For portfolio credit risk, the relation between PD and LGD is likely to be even more important than for models intended to price instruments issued by a single firm (firm-level models). The fact that portfolio losses depend upon both quantities and the fact that the periods when default is more likely to happen may also be the periods when recovery is lower suggest caution when using naive models to establish bank reserves and related precautionary measures.

APPENDIX A

Proof of Theorem 2.6 We fix $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{0\leq t\leq T})$ and an MPP, $\hat{\mu}$, with a compensator given by $\hat{\nu}(\mathrm{d}t, \mathrm{d}q) = m_t(\mathrm{d}q)\mathrm{d}t$. As before, $\mathcal{G}_t^W = \mathcal{G}^W \vee \mathcal{F}_t^{\mu}$. Since $M_t(x, \mathrm{d}q)$ is absolutely continuous with respect to $m_t(\mathrm{d}q)$ on $\mathcal{E}$, according to the Radon-Nikodym Theorem, for every t there exists a $\mathcal{E} \times \mathcal{G}^W$-measurable non-negative function $\varphi_t(q, x)$, $\varphi : E \times R_+ \to R_+$, such that:

$$\begin{aligned} M(t, Ax) &= \int_A \varphi(t, q, x) m(t, \mathrm{d}q), \text{ for all } A \in \mathcal{E} \quad \text{or} \\ M(t, \mathrm{d}q, x) &= \varphi(t, q, x) m(t, \mathrm{d}q) \end{aligned}$$

We define the process L_t as:

$$\begin{cases} \mathrm{d}L_t &= L_{t-} \int_E \{\varphi(t, q, X_t) - 1\} \{\hat{\mu}(\mathrm{d}t, \mathrm{d}q) - m_t(\mathrm{d}q)\mathrm{d}t\} \\ L_0 &= 1 \end{cases}$$

We notice that $\varphi(t, q, X_t) \in \mathcal{G}_0^W$. Define the new measure on $\mathcal{G}_t^W$, $0 \leq t \leq T$, as $\mathrm{d}\mathbb{Q} = L_t \mathrm{d}\mathbb{P}$. According to the Girsanov transformation, the $\mathbb{Q}$-compensator of the new process is exactly:

$$\begin{aligned} \nu(\mathrm{d}t, \mathrm{d}q) &= \hat{\nu}(\mathrm{d}t, \mathrm{d}q)(1 + \varphi_t(q, X_t) - 1) \\ &= \varphi_t(q, X_t) m_t(\mathrm{d}q)\mathrm{d}t = M_t(\mathrm{d}q, X_t)\mathrm{d}t \end{aligned}$$

First, we would like to show that the $\mathbb{Q}$-distribution of ν is the same as the $\mathbb{P}$-distribution. We note that $\mathcal{G}_0^W = \mathcal{G}^W$ and that $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\big|_{\mathcal{G}_0^W} = L_0 = 1$, thus, $\mathbb{P} = \mathbb{Q}$ on $\mathcal{G}_0^W$.

Second, we would like to show that $\mathbb{P}(\mu((s,t] \times B) = k \,\big|\, \mathcal{G}_s^W) = \frac{(\nu((s,t]\times B))^k}{k!} \mathrm{e}^{-\nu((s,t]\times B)}$, a.s., for $B \in E$. We prove it using characteristic functions.

Define the stochastic process $Y_t = \int_0^t \int_E q \hat{\mu}(\mathrm{d}s, \mathrm{d}q)$. Changing the measure, we obtain that $\mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{\mathrm{i}uY_t} | \mathcal{G}_0^W] = \mathbb{E}^{\mathbb{P}}[L_t \mathrm{e}^{\mathrm{i}uY_t} \mathcal{G}_0^W]$.

Define $Z_t = L_t e^{iuY_t}$. Then the dynamics of Z_t is:

$$\begin{aligned}
dZ_t =& L_t \int_E \{e^{iu(Y_{t-}+q)} - e^{iuY_{t-}}\}\mu(dt, dq) + L_{t-} e^{iuY_t} \int_E (\varphi(t, q, X_t) - 1)\{\hat{\mu}(dt, dq)\} \\
&- m_t(dq)dt + \int_E L_{t-}(\varphi(t, q, X_t) - 1)\{e^{iu(Y_{t-}+q)} - e^{iuY_{t-}}\}\hat{\mu}(dt, dq) \\
=& \int_E Z_{t-}\varphi(t, q, X_t) m_t(dq)(e^{iuq} - 1)dt + \int_E Z_{t-}(e^{iuq} - 1)\varphi(t, q, X_t)\tilde{\mu}(dt, dq) \\
&+ \int_E Z_{t-}(\varphi(t, q, X_t) - 1)\tilde{\mu}(dt, dq)
\end{aligned}$$

where $\tilde{\mu}(dt, dq) = \hat{\mu}(dt, dq) - m_t(dq)$. We notice also that $Z_0 = 1$. Then:

$$\begin{aligned}
Z_t &= 1 + \int_0^t \int_E Z_{s-}\varphi(s, q, X_s) m_t(dq)(e^{iuq} - 1)ds + \int_0^t \cdots \tilde{\mu}(ds, dq) \\
&= 1 + \int_0^t \int_E Z_{s-}(e^{iuq} - 1)M_s(dq, X_s)ds + \int_0^t \cdots \tilde{\mu}(ds, dq)
\end{aligned}$$

Denoting $\xi_t = \mathbb{E}^{\mathbb{P}}[Z_t \mathcal{G}_0^W]$, then $\xi_t = 1 + \int_0^t \int_E \xi_s(e^{iuq} - 1)M_s(dq, X_s)ds$. Since ξ_t does not depend on q and $M_s(dq, X_s)$ is $\mathcal{G}_0^W$-measurable, we have $\xi_t = e^{\int_0^t \int_E (e^{iuq}-1)M_s(dq,X_s)ds}$.

Note that $\nu(dt, dq, X_t) = M_t(dq, X_t)dt$ is $\mathcal{G}^W$ measurable. The final result follows from the fact that the characteristic function of the process $\bar{Y}_t = \int_0^t \int_E q\bar{\mu}(ds, dq)$, where $\bar{\mu}$ is an MPP with compensator $\bar{\nu}(t, dq)$, is given by $\mathbb{E}[e^{iu\bar{Y}_t}] = \exp\{\int_0^t \int_E (e^{iuq} - 1)\bar{\nu}(s, dq)\}$.

Lemma 1 *Consider a T-defaultable claim $\mathcal{X}$. For the purpose of computing expectations, and in particular its price at time $t \le T$, $\mathbb{E}_t^{\mathbb{Q}}[e^{\int_t^T r_s ds} V(T)\mathcal{X}]$, it is equivalent to use the following two dynamics for the remaining value process:*

$$\frac{dV(t)}{V(t-)} = -\int_0^1 q\mu(dt, dq), \quad V(t) = v \tag{11}$$

$$\frac{dV(t)}{V(t-)} = -q^e(t-, X_{t-})dN_t, \quad V(t) = v \tag{12}$$

where μ is a DSMPP with compensator $\nu(t, X_t) = \lambda(t, X_t)K(t, dq, X_t)dt$, N is a Cox process with intensity $\lambda(t, X_t)$, and we define $q^e(t, X_t) = \int_0^1 K(t, dq, X_t)$.

Proof Using the V dynamics in (11) we get:

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}[e^{-\int_t^T r_s ds} V(T)\mathcal{X} | \mathcal{F}_t] \\
&\quad = V(t)\mathbb{E}^{\mathbb{Q}}[e^{-\int_t^T r_s ds}\mathcal{X} | \mathcal{F}_t] - \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} \int_t^T \int_0^1 qV_{s-}\mu(dq, ds)\mathcal{X} \,\middle|\, \mathcal{F}_t \right]
\end{aligned}$$

$$
\begin{aligned}
&= V(t)\pi(t,\mathcal{X}) - \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r_s \mathrm{d}s}\int_t^T\int_0^1 qV_{s-}\mu(\mathrm{d}q,\mathrm{d}s)\mathcal{X}\,\middle|\,\mathcal{G}_t^W\right]\middle|\,\mathcal{F}_t\right]\\
&= V(t)\pi(t,\mathcal{X}) - \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r_s \mathrm{d}s}\int_t^T V_{s-}\left\{\int_0^1 qK(s,\mathrm{d}q,X_s)\right\}\lambda(s,X_s)\mathrm{d}s\mathcal{X}\,\middle|\,\mathcal{F}_t\right]\\
&= V(t)\pi(t,\mathcal{X}) - \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r_s \mathrm{d}s}\int_t^T V_{s-}q^e(s,X_s)\lambda(s,X_s)\mathrm{d}s\mathcal{X}\,\middle|\,\mathcal{F}_t\right]
\end{aligned}
$$

Using the V dynamics in (12) we get:

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{-\int_t^T r_s \mathrm{d}s}V(T)\mathcal{X}|\mathcal{F}_t]\\
&\quad= V(t)\mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{-\int_t^T r_s \mathrm{d}s}\mathcal{X}|\mathcal{F}_t] - \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r_s \mathrm{d}s}\int_t^T V_{s-}q^e(s,X_s)\mathrm{d}N(s)\mathcal{X}\,\middle|\,\mathcal{F}_t\right]\\
&\quad= V(t)\pi(t,\mathcal{X}) - \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r_s \mathrm{d}s}\int_t^T V_{s-}q^e(s,X_s)\mathrm{d}N(s)\mathcal{X}\,\middle|\,\mathcal{G}_t^W\right]\middle|\,\mathcal{F}_t\right]\\
&\quad= V(t)\pi(t,\mathcal{X}) - \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r_s \mathrm{d}s}\mathbb{E}^{\mathbb{Q}}\left[\int_t^T V_{s-}q^e(s,X_s)\mathrm{d}N(s)\mathcal{X}\,\middle|\,\mathcal{G}_t^W\right]\middle|\,\mathcal{F}_t\right]\\
&\quad= V(t)\pi(t,\mathcal{X}) - \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r_s \mathrm{d}s}\int_t^T V_{s-}q^e(s,X_s)\lambda(s,X_s)\mathrm{d}s\mathcal{X}\,\middle|\,\mathcal{F}_t\right]
\end{aligned}
$$

The results follow from comparing the final expressions on both cases. □

Proof of Proposition 2.8 The time t price of the defaultable zero-coupon bond with maturity T is equal to $\bar{p}(t,T) = \mathbb{E}_t^{\mathbb{Q}}[\mathrm{e}^{-\int_t^T r_s \mathrm{d}s}V(T)\mathcal{F}_t]$, where $V(T)$ is the residual of the face value after multiple defaults up to time T.

Making use of Lemma Appendix A1, instead of $\frac{\mathrm{d}V(t)}{V(t-)} = -\int_0^1 q\mu(\mathrm{d}t,\mathrm{d}q)$ with our DSMPP μ (these dynamics follow directly from (2)), we use $\frac{\mathrm{d}V(t)}{V(t-)} = -q^e(t-,X_{t-})\mathrm{d}N_t$, where N is the Cox process with intensity $\lambda(t,X_t)$.

For every fixed t, define $Z(u)$ as follows: $Z(u) = \mathrm{e}^{\int_t^u q^e(s,X_s)\lambda(s,X_s)\mathrm{d}s}V(u)$. We note, then, that the dynamics of $Z(u)$ take the form:

$$
\mathrm{d}Z(u) = -Z_{u-}q^e(u-,X_{u-})\{\mathrm{d}N_u - \lambda(u,X_u)\mathrm{d}u\}, \qquad u \geq t, \quad t\text{-fixed}
$$

and $Z(u)$ is a $\mathbb{Q}$-martingale conditional on the filtration $\mathcal{F}_t^W$. Thus, $\mathbb{E}^{\mathbb{Q}}[Z(T)|\mathcal{F}_t^W] = Z(t)$.

The price of a defaultable bond then can be found as:

$$
\begin{aligned}
\bar{p}(t,T) &= \mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{-\int_t^T r_s \mathrm{d}s}V(T)|\mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{-\int_t^T r_s \mathrm{d}s}\mathrm{e}^{-\int_t^T q^e(s,X_s)\lambda(s,X_s)\mathrm{d}s}Z(T)|\mathcal{F}_t]\\
&= \mathbb{E}^{\mathbb{Q}}[\mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{-\int_t^T r_s \mathrm{d}s}\mathrm{e}^{-\int_t^T q^e(s,X_s)\lambda(s,X_s)\mathrm{d}s}Z(T)|\mathcal{G}_t^W]|\mathcal{F}_t]\\
&= \mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{-\int_t^T r_s \mathrm{d}s}\mathrm{e}^{-\int_t^T q^e(s,X_s)\lambda(s,X_s)\mathrm{d}s}\mathbb{E}^{\mathbb{Q}}[Z(T)|\mathcal{G}_t^W]|\mathcal{F}_t]
\end{aligned}
$$

$$= \mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{-\int_t^T r_s \mathrm{d}s} \mathrm{e}^{-\int_t^T q^e(s,X_s)\lambda(s,X_s)\mathrm{d}s} Z(t)|\mathcal{F}_t]$$
$$= V(t)\mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{-\int_t^T \{r(s)+q^e(s,X_s)\lambda(s,X_s)\}\mathrm{d}s}|\mathcal{F}_t]$$

Using the basic relation between defaultable bond prices and defaultable forward rates (see Definition 2.1), we get:

$$\bar{f}(t,T) = \frac{\mathbb{E}_t^{\mathbb{Q}}[\{r(T)+\lambda(T,X_T)q^e(T,X_T)\}\,\mathrm{e}^{-\int_t^T \{r(s)+\lambda(s,X_s)q^e(s,X_s)\}\mathrm{d}s}]}{\mathbb{E}_t^{\mathbb{Q}}[\mathrm{e}^{-\int_t^T \{r(s)+\lambda(s,X_s)q^e(s,X_s)\}\mathrm{d}s}]}$$

Finally using $\bar{f}(t,t) = \bar{r}(t)$ in the above expression, we obtain $\bar{r}(t, r_t, X_t) = r(t) + q^e(t, X_t)\lambda(t, X_t)$.

The result follow from $s(t) = \bar{r}(t) - r(t)$ and $s(t,T) = \bar{f}(t,T) - f(t,T)$.

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